FROM PROCEDURAL ANALOGY TO UNDERSTANDING

NEIL HALL
Faculty of Education
University of Wollongong

BACKGROUND

In surveying the literature in mathematics education there seems to be a significant lack of success in explaining the relationship between the use of concrete embodiments in teaching mathematics and learning mathematical concepts and procedures. A colleague and I have developed a theory which has the potential to explain the circumstances under which the use of concrete materials produce efficient learning of mathematical concepts and skills (Ohlsson and Hall, 1990). The theory relies heavily on recent developments in mathematics education and cognitive science.

PROCEDURAL ANALOGY THEORY

The procedural analogy theory described briefly here is a theory of instruction in arithmetic. In addition to the original publication concerning this theory (Ohlsson and Hall, 1990), aspects of the theory have been presented elsewhere (Hall, 1990 and 1991). This theory is applicable to the use of concrete teaching materials by teachers who aim to increase learners' arithmetic skills and understandings beyond what would be the case without learners using such materials. That the use of these concrete materials may be seen as an attempt to better represent the abstractness of arithmetic facts and operations is hardly a new idea, but the empirical data supporting such notions is inconclusive.

The *raison d'etre* for concrete materials is to assist learners in internalising the mathematical concepts and skills represented by these materials. In cognitive science terminology, the materials assist the learning of declarative and procedural knowledge. The procedural analogy theory describes how concrete materials allow this declarative and procedural knowledge to be developed to the required target behaviour. Simplification, procedural analogy and symbolism lead finally to automatic responses.

The theory predicts that the pedagogical usefulness of an embodiment is a function of the degree of similarity of the procedure for the embodiment to the procedure for the initial symbolic representation, and argues that this relationship can be quantified. That is, this theory describes specific ways in which teachers may use concrete materials so as to increase the effectiveness of intended learning outcomes.

The four operations in algorithm format constitute a large portion of school children's mathematical experiences, certainly up to the beginning of secondary school. Yet we know that the completion of whole number algorithms from the learners' points of view is problematic. For example, subtracting one number from another is deceptively simple, provided you know how to do it. We will all be familiar with the range of incorrect approaches and the inventiveness of learners in creating incorrect solutions to algorithms.

If we consider the subtraction shown below on the left hand side of Figure 1, we expect learners to do something similar to what is written on the right hand side of the table.

5 4 2 - 2 6 3 2 7 9	3 from 2, cannot. Cross out 4, write 3. Write 1 beside 2. 3 from 12 is 9. 6 from 3, cannot. Cross out 5, write 4. Write 1 beside 3. 6 from 13 is 7. 2 from 4 is 2.

Figure 1: Simple subtraction

Table 1 shows that there are three or perhaps five steps in solving this algorithm. And this is true for an expert in subtraction algorithms. Unfortunately it is not true of learners, and may barely be true of many primary school teachers. For the novice, this simple subtraction is more likely to involve the steps shown in Figure 2.

5 4 2 - 2 6 3 2 7 9	0.0 542 - 263 (recognise question) 1.0 Process units 1.1 Take away 3 from 2 (cannot) 1.1.1 Trade for more units 1.1.2 Recall 4 - 1 = 3 1.1.3 Cross out 4, write 3 1.1.4 Write 1 next to 2 1.1.5 Recall this is 12 1.2 Take 3 from 12 1.3 Recall 12 - 3 = 9 1.4 Record 9 in answer space 2.0 Process tens 2.1 Take away 6 from 3 (cannot) 2.1.1 Trade for more tens 2.1.2 Recall 5 - 1 = 4 2.1.3 Cross out 5, write 4 2.1.4 Write 1 next to 3 2.1.5 Recall this is 13 2.2 Take 6 from 13 2.3 Recall 13 - 6 = 7 2.4 Record 7 in answer space 3.0 Process hundreds 3.1 Take 2 from 4 3.2 Recall 4 - 2 = 2 3.3 Record 2 in answer space 4.0 Read answer

Figure 2: 'simple' subtraction

Figure 2 shows that for the novice there may be 26 steps in solving this algorithm. Certainly over time chunking of steps would take place, as would the development of a more automatic response, so the actual number of steps is likely to decrease. On the other hand, those learners whose recall of number facts requires some calculation beyond quick reference to short term memory would increase the initial number of steps. There are two points that I want to make here. The first is that for novices, simple subtractions are anything but simple, they are complex multi-step operations with numerous opportunities for error. The second point relates to the value of the procedural analogy theory in drawing the teacher's attention to the need to cover every aspect of the algorithm to be taught.

MAB procedure	**Target procedure**
0.0 542 - 263	0.0 542 - 263
0.1 Subtract 263 from 5H, 4T, 2U	
1.0 Process units	1.0 Process units
1.1 Take 3U from 2U (cannot)	1.1 Take 3 from 2 (cannot)
1.1.1 Trade for more units	1.1.1 Trade for more units
1.1.2 Move 1L from 4L to bank bring back 10U	1.1.2 Recall 4 - 1 = 3
	1.1.3 Cross out 4, write 3
1.1.3 Join 10U and 2U	1.1.4 Write 1 next to 2
1.1.4 Recall 10U + 2U = 12U	1.1.5 Recall this is 12
1.2 Take 3U from 12U	1.2 Take 3 from 12
1.3 Recall 12U - 3U = 9U	1.3 Recall 12 - 3 = 9
1.4 Record answer, 9U in answer space	1.4 Record 9 in answer space
2.0 Process tens	2.0 Process tens
2.1 Take 6T from 3T (cannot)	2.1 Take 6 from 3 (cannot)
2.1.1 Trade for more longs	2.1.1 Trade for more tens
2.1.2 Move 1H from 5H to bank bring back 10T	2.1.2 Recall 5 - 1 = 4
	2.1.3 Cross out 5, write 4
2.1.3 Join 10T and 3T	2.1.4 Write 1 next to 3
2.1.4 Recall 10T + 3T = 13T	2.1.5 Recall this is 13
2.2 Take 6T from 13T	2.2 Take 6 from 13
2.3 Recall 13T - 6T = 7T	2.3 Recall 13 - 6 = 7
2.4 Record answer, 7T in answer space	2.4 Record 7 in answer space
3.0 Process hundreds	3.0 Process hundreds
3.1 Take 2H from 4H	3.1 Take 2 from 4
3.2 Recall 4H - 2H = 2H	3.2 Recall 4 - 2 = 2
3.3 Record answer, 2H in answer space	3.3 Record 2 in answer space
4.0 Read answer (2H 7T 9U)	4.0 Read answer (279)

Figure 3: procedural analogy

The basis of the procedural analogy theory rests on the use of concrete materials to draw a parallel between the numbers represented by these materials and the arithmetic processes represented by operations on these materials. That is, while materials may be used in a wide range of ways, there are ways that are more effective than others because they more closely resemble the target behaviour that we want learners to use in the solving of algorithms.

Figure 3 shows one use of Multibase Arithmetic Block (MAB) materials and the target algorithm that is developed from this material. The steps emphasised both in the use of MAB materials and in the target algorithm are not unique, and must be developed by the teacher. Once the teacher has decided on the target behaviour, an effective sequence can be developed for the concrete material.

The procedural analogy theory argues that the closer the relationship between the concrete procedure and the target procedure, the higher the analogy between the two sets of steps, and so the more effective will be the learning outcomes. The procedural analogy theory uses an isomorphism index ($I_{1,2}$) as a measure of analogy between the two sets of steps. The index is given by the formula

$$I_{1,2} = \frac{(N_1 + N_2 - 2) - (D_1 + D_2)}{N_1 + N_2 - 2}$$

where N_1 is the number of steps in the first procedure, N_2 the number of steps in the second procedure, D_1 the number of steps in the first procedure but not in the second, and D_2 the number in the second procedure but not in the first. In Table 3, $N_1 = 25$, $N_2 = 26$, $D_1 = 5$ and $D_2 = 6$ giving a value for I of 0.78. This is a relatively high value which is quite difficult to increase in this algorithm. This simply reflects the reality that there are differences between using concrete materials and in writing algorithms. At the same time, slight variations in the steps will lead to a lower isomorphism index. The theory maintains that the higher the I value the more effective will be the value of the concrete materials, and the greater the learning outcomes. That is, the procedural analogy theory provides a method of measuring likely pedagogical success, and one allowing alterations in teaching steps to be assessed prior to teaching.

Application of the theory requires the following steps:

(a) select the arithmetic topic;

(b) identify a procedure which an expert would use to answer questions in this topic;

(c) develop a detailed procedure as target behaviour for learners;

(d) identify suitable concrete materials to support learning this topic;

(e) develop a detailed procedure for the use of these materials;

(f) use the Isomorphism Index to contrast alternatives with concrete materials, and with the target behaviour;

(g) use the Isomorphism Index to identify the best set of teaching procedures;

(h) teach the topic.

RESEARCH DATA

A small scale research project was designed to test this procedural analogy theory. This pilot study involved teaching subtraction to small groups (n=5) of primary school children, selected by virtue of one school's accessibility. Three groups of children were established: experimental group 1 used MAB materials, expanded notation and the target subtraction algorithm. Experimental group 2 used MAB materials and the target algorithm, while the control group used the target algorithm only. The results provided some support for this theory, but none were statistically significant (Kelly, 1990).

I am presently involved in a larger scale research project involving two schools, 120 children, five teachers and three teaching approaches to subtraction using MAB materials. Data gathering is still underway, so no meaningful statistical analyses can be reported here.

DISCUSSION

The present research seeks to verify the procedural analogy theory described here. There are clearly many aspects that could be discussed: the theory itself, the research approach, the data I have collected and ways in which they will be analysed. But I have a particular question that needs answering.

Even though the larger part of this paper has been given over to describing the theory, the description is in the way of a precursor to this discussion. I have some ideological difficulties with this procedural theory: mathematics has to be more than procedures. In particular, assuming that the present theory has some veracity, I want to extend the present theory so as to investigate the extent to which 'understanding' takes place in the learner, and to gain some measure of the efficiency of learners' cognitive networks of mathematical skills and concepts. That is, it is certainly educationally worthwhile to be able to help learners solve algorithms more efficiently, and that is no mean feat. But contemporary mathematics education is less about correct answers and more about understanding and constructing meanings. If this procedural theory is correct, and for argument's sake let me assume it is, then some important questions arise. For example, which steps in the concrete materials sequence or in the target procedure lead to understanding? And how can we gain insights into and prove the existence of such understandings? Further, how does this increased understanding influence learners' networks of cognitive structures, and how can we prove it?

This paper ends then with a list of research questions generated from the procedural analogy theory described here: questions about the procedural analogy theory, about pedagogy, about understanding and about cognitive structures. Such questions include

> How generalisable is the procedural analogy theory, in terms of mathematics topics and ages of learners?
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> Does the procedural analogy theory lead to increases in understandings?
>
> Do teaching approaches developed through the procedural analogy theory, in contrast to other teaching approaches, increase the likelihood of transfer between topics, and increase learners' problem solving abilities?

Does application of this procedural analogy theory encourage learners to develop a richly connected network of cognitive structures?

Is the cognitive development taking place through the application of this procedural analogy theory superior to the developments using other teaching approaches?

What instruments will give researchers a clearer picture of changes in learners' understandings and cognitive structures?

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